POLAR COORDINATES: WHAT THEY ARE AND HOW TO USE THEM MAHNOUD M.ALAMRE

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1- 1) Coordinate system:

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x,y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. .This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. Coordinates in this form are called polar coordinates.

The usual Cartesian coordinate system can be quite difficult to use in certain situations. Some of the most common situations when Cartesian coordinates are difficult to employ involve those in which circular, cylindrical, Double Integrals or spherical symmetry is present. For these situations it is often more convenient to use a different coordinate system as Polar Coordinates.

The coordinate system as a rule. The most basic question is: What is a coordinate system? The answer is so important that I am going to state it in bold font:

A coordinate system is a rule for mapping pairs of numbers to points in the plane.

This may not make much sense to you right now, but you'll see what I mean shortly below when we discuss the x,y and the polar coordinate systems. I do want to emphasize two things:

A coordinate system is not just a set of axes, it is a set of rules for mapping a pair of numbers onto a point in the plane.

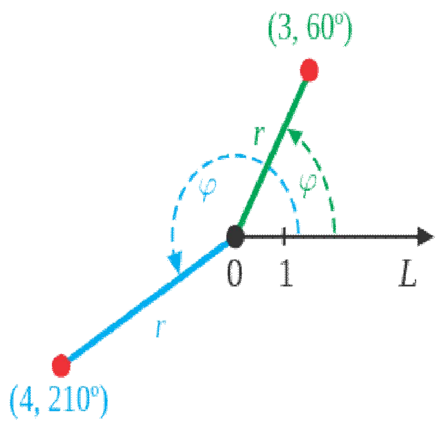
2-1) Different coordinate systems correspond to different rules:

The polar coordinate system has rules that are different than the rules of the X,Y coordinate system. Other coordinate systems have yet other rules. Learning a new coordinate system comes down to understanding its rules. In this search we will learn how to use polar coordinates to evaluate certain integrals. Before we look at the details of this search, let's recall a few facts about polar coordinates. Keep this in mind as you read the rest of this search.

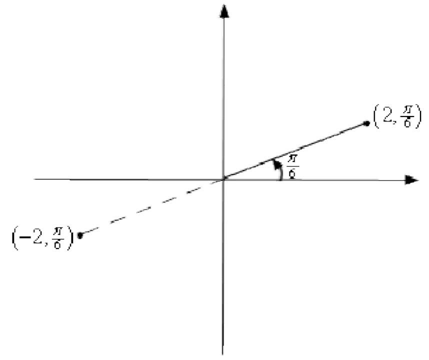
3-1) polar coordinate system:

In mathematics, the **polar coordinate system** is a two-dimensional coordinate system in which each point(p) on a plane is determined by a distance from a reference point (O) and an angle (φ) from a reference direction. The reference point O (analogous to the origin of a Cartesian system) is called the **pole**, and the ray from the pole in the reference direction is the **polar axis L**. The distance from the pole is called the **radial coordinate** or **radius**, and the angle is called the **angular coordinate**, **polar angle**.The radial coordinate is often denoted by r or ρ, and the angular coordinate by φ, θ, or t. The angular coordinate is specified as φ by ISO standard 31-1.Angles in polar notation are generally expressed in either degrees or radians (2π rad being equal to $360°$). Degrees are traditionally used in navigation, surveying, and many applied disciplines, while radians are more common in mathematics and mathematical physics. In many contexts, a positive angular coordinate means that the angle φ is measured counterclockwise from the axis. In mathematical literature, the polar axis is often drawn horizontal and pointing to the right.

Points in the polar coordinate system with pole *O* and polar axis *L*. The point with radial coordinate 3 and angular coordinate 60 degrees or (3,60°). The point (4,210°).



The above discussion may lead one to think that *r* must be a positive number. However, we also allow *r* to be negative. Below is a sketch of the two points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$.

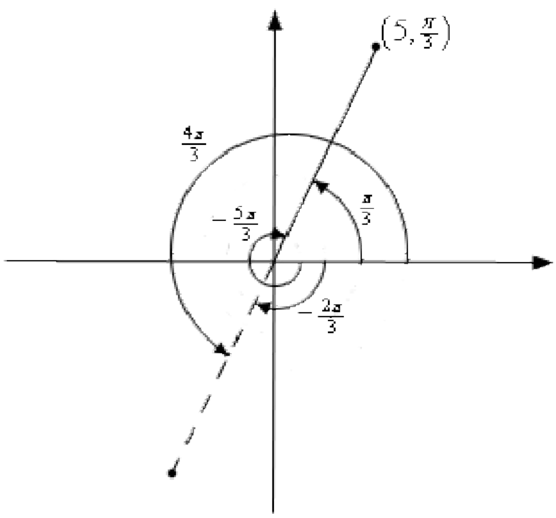


From this sketch we can see that if *r* is positive the point will be in the same quadrant as θ. On the other hand if *r* is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates$\left(-2, \frac{\pi}{6}\right)$. describe the same point as the coordinates$\left(2, \frac{7\pi}{6}\right)$ do. The coordinates$\left(2, \frac{7\pi}{6}\right)$tells us to rotate an angle of $\left(\frac{7\pi}{6}\right)$ from the positive *x*–axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point.

For instance, the following four points are all coordinates for the same point.

$$\left(5,\frac{\pi}{3}\right)=\left(5,-\frac{5\pi}{3}\right)=\left(-5,\frac{4\pi}{3}\right)=\left(-5,-\frac{2\pi}{3}\right)$$

Here is a sketch of the angles used in these four sets of coordinates.



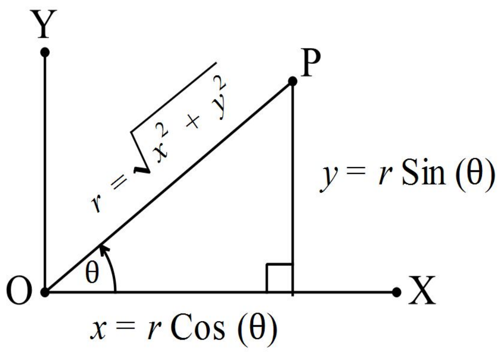
$4-1$)Uniqueness of polar coordinates:

Adding any number of full turns $(360°)$ to the angular coordinate does not change the corresponding direction. Also, a negative radial coordinate is best interpreted as the corresponding positive distance measured in the opposite direction. Therefore, the same point can be expressed with an infinite number of different polar coordinates (r, φ ± n×360°) or (−r, φ ± (2n + 1)180°), where n is any integer. Moreover, the pole itself can be expressed as (0, □) for any angle □. Where a unique representation is needed for any point, it is usual to limit r to non−negativnumbers (r ≥ 0) and □ to the interval [0, 360°) or (−180°, 180°] (in radians, [0, 2π) or (−π, π]). One must also choose a unique azimuth for the pole, e.g., □ = 0.

5 − 1) Relation between Polar and Cartesian coordinates:

The polar coordinates *r* and φ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$$x = r \operatorname{Cos}(\theta)$$
$$y = r \operatorname{Sin}(\theta) \quad \rightarrow (1)$$



The Cartesian coordinates x and y can be converted to polar coordinates r and □ with r ≥0 and □ in the interval (−π, π] by:

$r = \sqrt{x^2 + y^2}$ (As in the Pythagorean Theorem or the Euclidean norm).

And $\varphi = \arctan\left(\frac{y}{x}\right)$

Where the arctangent function defined as:

$$\varphi = \arctan\left(\frac{y}{x}\right) = \begin{cases} \arctan\left(\frac{x}{y}\right) & if\ x > 0 \\ \arctan\left(\frac{x}{y}\right) + \pi & if\ x < 0\ and\ y \geq 0 \\ \arctan\left(\frac{x}{y}\right) - \pi & if\ x < 0\ and\ y < 0 \\ \frac{\pi}{2} & if\ x = 0\ and\ y > 0 \\ -\frac{\pi}{2} & if\ x = 0\ and\ y < 0 \\ undefined & if\ x = 0\ and\ y = 0 \end{cases} \quad (2)$$

This following formulas used to converting the points and equations from Cartesian coordinates to polar coordinates.

Example:

Polar Coordinates Cartesian Coordinates

Point $P\ (2, 30^o) P\ (2Cos(30^o), 2Sin(3^o) = P(\sqrt{3}, 1)$

Polar Equation Cartesian Equation

$$r^2 Cos\ \theta - r^2 Sin\ \theta = 1 \quad x^2 - y^2 = 1$$

$$r^2 Cos\ \theta Sin\ \theta = 4 \quad xy = 4$$

6 – 1) Polar equation of a curve:

The equation defining an algebraic curve expressed in polar coordinates is known as a polar equation. In many cases, such an equation can simply be specified by defining r as a function of φ. The resulting curve then consists of points of the form $(r = f(\varphi), \varphi)$ and can be regarded as the graph of the polar function r.

1 – 2)Generalization:

Using Cartesian coordinates, an infinitesimal area element can be calculated as

dA = dx dy. The substitution rule for multiple integrals states that, when using other coordinates, the Jacobean determinant of the coordinate conversion formula has to be considered:

$$J = \det\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} Cos\,\varphi & -rSin\,\varphi \\ Sin\,\varphi & r\,Cos\,\varphi \end{vmatrix} = r$$

Hence, an area element in polar coordinates can be written as

$$dA = dx\,dy = J\,dr\,d\varphi = r\,dr\,d\varphi$$

Now, a function that is given in polar coordinates can be integrated as follows:

$$\oiint_R f(x,y)dA = \int_a^b \int_\varphi^{r(\varphi)} f(r,\varphi)\, r\, dr\, d\varphi$$

Here, *R* is the same region as above, namely, the region enclosed by a curve *r(φ)* and the rays φ = a and φ = b.

2 −2) Double Integrals:

in this article we define the integral of the function f(x,y)of two variables over rectangular region in the Cartesian coordinates, we will used the formulas (1) and (2) to converting the function from Cartesian coordinates to polar coordinates.

3−2− Application:

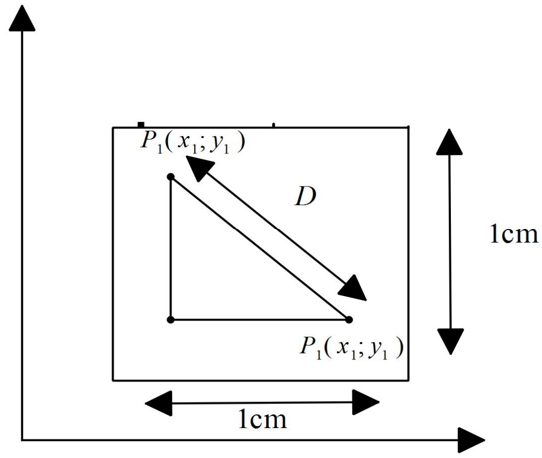
Example:

We consider a square whose side length is equalto one, we pick two points at random. The equation is: Given all the different ways that we can pick two points at random: What will be the average distance between the two points?

Place the unit square in the Cartesian Plan, we can give each of the two points,

$P_1(x_1, y_1)$ And $P_2(x_2, y_2)$, therefore we can use (The Pythagorean Theorem) and came up with the distance formula:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



So we have:

1- Method (1): Estimate with random number.

2 – Method (2): Exact answer with integral.

1- Method (1): Estimate with random number: Numerical Solution:

The first method we`ll present is we can generate random numbers for our coordinates, come up with a distance and take the average over many trials.

Let`s first come up with an estimate,let's estimate our answer numerically, we went to generate for coordinates (x) and (y) as in table (1), to estimate our answer numerically, we want to generate four coordinates the (x) and (y) coordinates for the first point and (x) and (y) for the second point, in each trial we`ll compute the distance between these two points, we`ll then take the average over many trials, we can generate a random coordinate between (0) and (1) using the random function which generates a random number between (0) and (1), in each rail we'll take the distance by:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This will give us an distance in one trial, we'll copy this down over (100) times trials, and now we can figure out the average by taking the average over these trials (**TABLE** (1)).

So we'll take the average distances over (100) trials and this will give us an estimate of point we figured out our answer is about (0.533). But with is it the exactly answer? Let's figure out using **Integrals** in the second method.

TABLE (1): Estimate with random number for two points (NumericalSolution)

TABLE (1): Estimate with random number for two points (Numerical Solution)

				Average	0.533					
x1	y1	x2	y2	Distance		x1	y1	x2	y2	Distance
0.647386	0.449794	0.998624	0.694276	0.4279486		0.494021	0.119161	0.895425	0.812013	0.800731
0.831948	0.596045	0.161946	0.732125	0.6836815		0.759404	0.740853	0.34673	0.504492	0.47557
0.822404	0.010577	0.474526	0.743397	0.8111991		0.42209	0.913862	0.341315	0.045012	0.872596
0.863538	0.344046	0.201657	0.484509	0.6766205		0.928502	0.97592	0.392771	0.260933	0.893428
0.632546	0.781822	0.542415	0.627121	0.1790411		0.59378	0.107169	0.322979	0.336567	0.354904
0.63157	0.689904	0.644592	0.579529	0.1111402		0.43289	0.806896	0.812848	0.411192	0.548589
0.452292	0.725538	0.895191	0.44562	0.5239404		0.331728	0.495844	0.959397	0.46709	0.628327
0.790206	0.677692	0.268055	0.707253	0.5229875		0.135899	0.557123	0.545775	0.736977	0.4476
0.530711	0.164009	0.509074	0.020361	0.1452687		0.875883	0.213765	0.539779	0.565924	0.486808
0.126701	0.613907	0.052762	0.278095	0.3438547		0.079563	0.283693	0.816082	0.770596	0.882912
0.47063	0.985928	0.40071	0.089328	0.8993223		0.953192	0.836907	0.471187	0.519768	0.57698
0.363537	0.191078	0.966014	0.062028	0.6161434		0.073746	0.045971	0.281799	0.351033	0.369255
0.938175	0.070915	0.01211	0.468976	1.0079919		0.816719	0.092114	0.46638	0.172002	0.359332
0.438588	0.005014	0.560371	0.498063	0.5078667		0.392769	0.263892	0.950669	0.933302	0.871413
0.830538	0.852923	0.196719	0.72514	0.6465715		0.087273	0.058998	0.887753	0.721456	1.039048
0.830436	0.838528	0.560253	0.698238	0.3044343		0.124649	0.76101	0.075139	0.928974	0.175109
0.507234	0.894154	0.318843	0.884393	0.1886442		0.781104	0.472265	0.287033	0.53306	0.497797
0.525577	0.058324	0.738113	0.985181	0.9509125		0.675806	0.975968	0.810766	0.936741	0.140546
0.292476	0.506198	0.986245	0.776069	0.7444091		0.339114	0.205327	0.903859	0.613305	0.696694
0.275874	0.086471	0.247697	0.876969	0.7909996		0.992882	0.012008	0.912258	0.710638	0.703266
0.077026	0.984172	0.716312	0.351014	0.8997644		0.569325	0.153213	0.933812	0.565539	0.550329
0.861731	0.265058	0.211485	0.979738	0.966223		0.292302	0.751815	0.727351	0.369654	0.579063
0.718643	0.877425	0.435864	0.4219	0.53616		0.886072	0.418506	0.74401	0.804303	0.411121
0.702063	0.913298	0.006558	0.605125	0.7607223		0.489847	0.278549	0.738192	0.752074	0.534698
0.871671	0.511416	0.438243	0.876667	0.5668046		0.98283	0.778675	0.012657	0.949264	0.985056
0.531775	0.145273	0.095387	0.974683	0.937206		0.197215	0.125711	0.70277	0.308649	0.537636
0.580829	0.43584	0.933083	0.334791	0.3664615		0.978529	0.500983	0.159475	0.116081	0.904986
0.380964	0.273536	0.942494	0.260519	0.5616804		0.278979	0.65317	0.447762	0.857482	0.265012
0.953783	0.342041	0.861148	0.345139	0.092687		0.547326	0.775146	0.865985	0.276552	0.591727
0.002029	0.339797	0.76311	0.300855	0.762076		0.757451	0.509014	0.601503	0.570802	0.167742
0.808073	0.103186	0.994726	0.808323	0.7294227		0.057745	0.698342	0.216352	0.433032	0.309105
0.202256	0.340846	0.4927	0.061018	0.403313		0.184563	0.583977	0.150262	0.373391	0.213361
0.902569	0.909873	0.088336	0.695465	0.8419901		0.062148	0.045572	0.932037	0.699743	1.088414
0.218997	0.911441	0.57943	0.821146	0.3715714		0.374156	0.807899	0.748792	0.458699	0.512145
0.167229	0.92316	0.253055	0.087182	0.8403719		0.562145	0.139989	0.168065	0.060077	0.402101
0.702993	0.510818	0.566549	0.678549	0.2162191		0.708584	0.765138	0.739456	0.946612	0.184081
0.553881	0.480103	0.443534	0.785297	0.3245301		0.099119	0.896318	0.909978	0.683601	0.838297
0.111683	0.475223	0.760047	0.879813	0.7642435		0.575314	0.422142	0.752153	0.556057	0.221822
0.778614	0.204308	0.907222	0.454078	0.2809363		0.447903	0.811652	0.381118	0.891523	0.104113
0.398605	0.028398	0.087777	0.605894	0.6558318		0.41078	0.751396	0.482515	0.858953	0.129284
0.828115	0.891313	0.422953	0.530531	0.5425119		0.32568	0.839446	0.793731	0.061838	0.907605
0.666486	0.437176	0.173326	0.209065	0.5433604		0.033119	0.694104	0.250892	0.466582	0.314946
0.847015	0.404024	0.574088	0.472505	0.2813873		0.434881	0.629431	0.125155	0.755232	0.334299
0.870844	0.785349	0.621397	0.571109	0.3288203		0.039737	0.758176	0.915849	0.074797	1.111116
0.447611	0.203551	0.007101	0.184201	0.4409351		0.491446	0.369806	0.34363	0.052624	0.349934
0.879082	0.689212	0.396239	0.983315	0.5653624		0.126432	0.171592	0.803718	0.657994	0.833849
0.319456	0.739994	0.211007	0.654243	0.1382554		0.668205	0.156148	0.14213	0.276918	0.539759
0.487507	0.167354	0.384458	0.90932	0.7490879		0.539824	0.095277	0.65005	0.651573	0.567111

2 – Method (2): Exact answer with integral:

1− Integrate average distance

$$\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{(x-x)^2+(y-y)^2}\, dx_1 dx_2 dy_1 dy_2$$

2 − We have simplify into x and y distances

$$(x_2-x_1) = |\Delta_x| , (y_2-y_1) = |\Delta_y|$$

$$D = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(\Delta_x)^2+(\Delta_y)^2}$$

If u and v are uniform distributions (p.d.f is 1), then w= $|u-v|$ has triangular distribution (p.d.f is 2(1−w))

$$D = 4\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2+y^2}\,(1-x)(1-y)dx\,dy$$

3 – Change to polar coordinates:

$$x = r\,Cos\,\theta\ ,\qquad y = rSin\,\theta\ ,$$

$$0 \le \theta \le {}^{\pi}/_{4}\ \ and\ r = {}^{1}/_{Cos\,\theta} = Sec\,\theta$$

Then $$D = 4\int_0^{\pi/4} 2\int_0^{Sec\,\theta} f(r,\theta)\,rdr\,d\theta$$

$$D = 8\int_0^{\pi/4}\int_0^{Sec\,\theta} \sqrt{r^2cos^2\theta + r^2sin^2\theta}\ (1-r\cos\theta)(1-\sin\theta)r\,dr\,d\theta$$

$$D = 8\int_0^{\pi/4}\int_0^{Sec\,\theta} r\,(1-r\cos\theta)(1-\sin\theta)r\,dr\,d\theta$$

$$D = 8\int_0^{\pi/4}\int_0^{Sec\,\theta} \{r^2 - r^3\cos\theta - r^3\sin\theta + r^4\cos\theta\sin\theta\}\,dr\,d\theta$$

$$D = 8\int_0^{\pi/4}\{\frac{\sec^3\theta}{12} - \frac{\sec^3\theta\tan\theta}{20}\}\,d\theta$$

$$D = 8(\frac{\sec\theta\tan\theta + Ln|\sec\theta\tan\theta|}{24} - \frac{sec^3\,\theta}{60})_0^{\pi/4}\,d\theta$$

$$D = \frac{2+\sqrt{2}+5\,Ln(\sqrt{2}+1)}{15} = \cong 0.521.$$

3−3) Concluding and Remarks:

1− Evaluate the unsolvable integrals by transforming to Polar Coordinate System: The relationships between the Cartesian and Polar coordinates enable us to change any Cartesian equation into a polar equation for the same curve.

2− We can use Polar coordinate to find the Areas and Volumes of regions enclosed by graphs of Polar coordinate.

3− We can use Polar coordinate to find an important result in Statistics, the value of integral as:

$I = \int_0^{\infty}\{ e^{-x^2}\}\ d\theta$And$I = \int_0^{\infty}\{x^2 e^{-x^2}\}\ d\theta$

There are very important in statistics applications.

4− Polar coordinates are applied in the description of characteristics of microphones, as well as in navigation systems to locate positions on the polar graph, such as in maritime radar systems. The pickup patterns of a microphone are usually plotted using polar coordinates to indicate its sound reception.

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