On Propositional calculus (Mathematical Logic)
The Soundness and TheCompleteness of The Non-Formal Systems Deduction Systems (DSi, $1 \leq i \leq 4$)

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Abstract

The purpose of this paper use The writer the non Axiomatic logical systems (normal logical systems, the Axiomatic logical systems and the axiomatic logic) , Language and definitions, Operators, Inductive clause I and Inductive clause II, To proof of theorems of the non- formal systems DS_1, DS_2, DS_3, DS_4 will be presented

- The deduction system the DSi,$1 \leq i \leq 4$
- The soundness and completeness of the DSi,$1 \leq i \leq 4$
- Definition(soundness 1
- Definition a model
- The Completeness Theorem
- Theorem (Godel Completeness Theorem)

Introduction:

The propositional calculus is a branch of mathematical logic sometimes called propositional logic, it deals with the study of mathematical and logic ,it divides into two mains branches.

- Non Axiomatic logical systems (normal logical systems) .
- Axiomatic logical systems (the axiomatic logic).

In the study of non- Axiomatic logical systems we use a natural deduction system without axioms, which has an empty axiom set. to study and proof Thermos of the deduction systems DSi,$1 \leq i \leq 4$

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1. Language and definitions:

1-1 Atomic proposition: An atomic proposition is a sentence contains only one content either true or falls. The small letters of the alphabet (a,b,c...etc) standing as atomic proposition.

1-2 Operators: symbols denoting the following connectives (or logical operators): ¬, ∧, ∨, ⊃, ↔.

1-3 Parentheses: Left and right parentheses: (,) , { [(,)] }

1-4 Complex proposition: a complex proposition is a composition of more than one atomic proposition with some operators and parentheses, the capital letters of the alphabet (A , B, C) standing as complex proposition.

1-5 well formed formula (wff): A well formed formula (wff)is a set of complex propositions is recursively defined by the following rules:

- Basis: Letters of the alphabet (usually capitalized such as A, , B, ,C,D,etc.) or the Greek alphabet (χ , φ, ψ)are well-formed formulas wffs is recursively defined by the following rules:

- Inductive clause I: If φ is a wff, then ¬ φ is a wff.

- Inductive clause II: If φ and ψ are wffs, then (φ ∧ψ), (φ ∨ψ), (φ → ψ),and (φ ↔ ψ) are wffs.

1.6 Rules of inferences:

A rule of inference is a valid argument used to deduct a new wff from a previous wffthe following are some of rules of inferences:

R_1 : Simplification $p \wedge q \vdash p$ Simp

R_2 : Com mutative $p \wedge q \vdash q \wedge p$ Com

R_3: Conj unction $p , q \vdash p \wedge q$ Conj

1.7 Rules of manipulation:

Proposition (1.1) :If *A* and *A* →*B* are tautologies, then so is *B*.

Proof. Suppose that *A*and *A* →*B* are tautologies, and that *B* is not. Then there is an assignment of truth values to the statement letters appearing in *A* or in *B* which gives *B* the value F. But it must give *A* the value T since *A* is a tautology, and so it gives *A* →*B* the value F. This contradicts the assumption that *A*→*B* is a tautology. Hence *B*must be a tautology. Rules of manipulation and substitution.

1.8 Rules of substitution:

Proposition (1.2): Let A be a wff in which the statement letters P_1 , P_2 ,........, P_n appear, and let A_1 , A_2 ,........, A_n be any wffs. If A is a tautology then the statement form B , obtained from A by replacing each occurrence of $P_i$by A_i ($1 \leq i \leq n$)throughout, is a tautology also, i.e. substitution in a tautology yields a tautology.

Proof: Let A be a tautology and let P_1 , P_2 ,........, $P_n$be the statement letters appearing in A . Let A_1 , A_2 ,........, $A_n$be any statement forms. Assign any truth values to the statement letters which appear in $A_1, A_2, \ldots, A_n$ The truth value that *B* now takes is the same as that which *A* would have taken if the values which $A_1, A_2, \ldots, A_n$ take had been assigned to $P_1, P_2, \ldots, P_n$ respectively, namely T. Hence *B* takes value T under any assignment of truth values, i.e. *B* is a tautology.

Now consider the statement form $((A \wedge A) \rightarrow B)$. $(A \wedge A)$, which appears in this form, is logically equivalent to *A* (since $((A \wedge A) \equiv A)$ is a tautology). If wereplace $(A \wedge A)$ by *A*, we get $(A \rightarrow B)$. Now $(A \rightarrow B)$ is logically equivalent to $((A \wedge A) \rightarrow B)$. Again this is an instance of general proposition substitution

1.9 <u>A proof:</u>

We will use a natural deduction system, which has no axioms; or, equivalently, which has an empty axiom set. Derivations using our calculus will be laid out in the form of a list of numbered lines, with a single wff and a justification on each line. Any given wff considered to be assumptions and written in the top of the proof . The conclusion will be on the last line. A derivation will be considered complete if every line follows from previous ones by correct application of a rule

Theorem:

The last wff in the proof called a theorem .

2.1 <u>The deduction system DS1</u>

In this section of this paper discussion and proofs of theorems of the non-formal systems DS_1, DS_2, DS_3, DS_4 will be presented.

2.2 Rules of inferences of DS_1:

1. $(A \wedge B) \vdash A$ *Simplification*
2. $(A \wedge B \vdash (B \wedge A)$ *Commutative*
3. $A, B \vdash (A \wedge B)$ *Conjunction*

Theorem2- 1-1: $A \wedge (B \supset C) \vdash A$

Proof

1) $A \wedge (B \supset C)$ assum.

2) A 1 ,simp.

$A \wedge (B \supset C) \vdash A$

Theorem2-1-2: $(B \vee C) \wedge E \vdash E$

1) 1. $(B \vee C) \wedge E$ assumption

2) 2. $E \wedge (B \vee C)$ 1, com.

3) E 2, simp.

$(B \vee C) \wedge E \vdash E$

Theorem 2-1-3: $C \wedge (D \wedge E) \vdash D$

Proof:

1) $C \wedge (D \wedge E)$ assumption

2) $(D \wedge E) \wedge C$ 1, com.

3) $D \wedge E$ 2, simp.

4) 4) D 3, simp.

$C \wedge (D \wedge E) \vdash D$

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Theorem 2-1-4 : $A \vee D$, $B \wedge C \vdash C \wedge (A \vee D)$

Proof :

1. $A \vee D$ assumption

2. $B \wedge C$ assumptio

3. $C \wedge B$ 2, com.

4. C 3, simp.

4. $C \wedge (A \vee D)$

5. 4,1, conj.

$A \vee D$, $B \wedge C \vdash C \wedge D$

Theorem2-1-5: $(A \wedge B) \wedge C \vdash B \wedge C$

Proof :

1. $(A \wedge B) \wedge C$ assumption

2. $A \wedge B$ 1, simp.

3. $C \wedge (A \wedge B)$ 1, com.

4. C 3, simp.

5. $B \wedge A$ 2, com.

6. B 5, simp.

7. $B \wedge C$ 4, 6, conj.

$(A \wedge B) \wedge C \vdash B \wedge C$

<u>The deduction system DS_2</u>

Rules of inferences of DS_2

1. $(A \vee B), \neg A \vdash B$ Disjunctions syllogism(DS)

2. $(A \vee B) \vdash (B \vee A)$ Commutative

3. $A \vdash (A \vee B)$ Addition

Theorem 2-2-1:-

$\neg B, A \vee B \vdash A$

Proof

1. $\neg B$ Assumption
2. $A \vee B$ Assumption
3. $B \vee A$ 2 , Com
4. A 3 , 1 ,DS

$\therefore \neg B, A \vee B \vdash A$

Theorem 2-2-2:-

$C \wedge D \vdash D \vee E$

Proof

1. $C \wedge D$ assumption
2. $D \wedge C$ 1 , Com
3. D 2 , Simp

4. D ∨ E 3 , add

∴ C ∧ D ├ D ∨ E

Theorem2-2- 3:-

(A∨ B) ∧¬B ├ A

Proof

1. (A ∨ B) ∧¬B assumption
2. A ∨ B 1,simp
3. ¬B ∧ (A ∨ B) 1 , Com
4. ¬B 3 , Simp
5. B ∨ A 2 , Com
6. A 5 , 4 , DS

∴ (A∨ B) ∧¬B ├ A

Theorem2-2- 4:-

¬(A ∨ B) , (C ⊃ D) ∨ (A ∨ B) , ¬D ├ (C ⊃ D) ∧ (E ∨¬D)

Proof

1. ¬(A ∨ B) assumption
2. (C ⊃ D) ∨ (A ∨ B) assumption
3. ¬D assumption
4. (A ∨ B) ∨ (C ⊃ D) 2 , Com
5. C ⊃ D 2 , 1 , DS
6. ¬D ∨ E 3 , Add
7. E ∨¬D 6 , Com
8. (C ⊃ D) ∧ (E ∨¬D) 5 , 7 , conj

∴ ¬(A ∨ B) , (C ⊃ D) ∨ (A ∨ B) , ¬D ├ (C ⊃ D) ∧ (E ∨¬D)

Theorem 2-2-5:-

¬(B ⊃C)∧A , (E ⊃ D) ∨ (B ⊃ C) ├ (D ∨ A) ∧ (E ⊃ D)

Proof

1. ¬(B ⊃ C) ∧ A assumption
2. (E ⊃ D) ∨ (B ⊃ C) assumption
3. ¬(B ⊃ C) 1 , Simp
4. A ∧¬(B ⊃ C) 1 , Com
5. A 4 , Simp
6. (B ⊃ C) ∨ (E ⊃ D) 2 , Com
7. E ⊃ D 6 , 3 , DS
8. A ∨ D 5 , Add
9. D ∨ A 8 , Com
10. (D ∨ A) ∧ (E ⊃ D) 9 , 7 , Conj

∴ ¬(B ⊃C)∧A , (E ⊃ D) ∨ (B ⊃ C) ├ (D ∨ A) ∧ (E ⊃ D)

The deduction system DS_3

Rules of inferences of DS_3

1(A ⊃ B) , A ├B Modus Ponnens (MP)

1. (A ⊃ B) , ¬B ├¬A Modus Tollens (MT)

Theorem 2-3-1:-

A⊃B , A ├B

Proof

1. A⊃B assumption
2. A assumption
3. B 1 , 2 , MP

∴ A⊃B , A ├B

Theorem 2-3-2:-

¬A⊃¬B ,¬¬B ├¬¬A

Proof

1. ¬A⊃¬B assumption
2. ¬¬B assumption
3. ¬¬A 1 , 2 , MT

∴¬A⊃¬B ,¬¬B ├¬¬A

Theorem 2-3-3 :-

A∧(A⊃B) ├B

Proof

1. A∧(A⊃B) assumption
2. A 1 ,Simp.
3. (A⊃B)∧A 1 , Com.
4. A⊃B 3 , Simp
5. B 2 , 4 , MP

∴ A∧(A⊃B) ├B

Theorem2-3-4:-

(A⊃B)∧(B⊃C) ,¬C ├¬A

Proof

1. (A⊃B)∧(B⊃C) assumption
2. ¬C assumption
3. A⊃B 1 , Simp
4. (B⊃C)∧(A⊃B) 1 , Com
5. B⊃C 4 , Simp
6. ¬B 2 , 5 , MT
7. ¬A 3 , 6 , MT

∴ (A⊃B)∧(B⊃C) ,¬C ├¬A

Theorem2-3-5 :-

(A⊃B)∧(B⊃C) , C⊃D , A ├D

Proof

1. (A⊃B)∧(B⊃C) assumption
2. C⊃D assumption
3. A assumption

4. $A \supset B$ 1, Simp
5. $(B \supset C) \wedge (A \supset B)$ 1, Com
6. $B \supset C$ 5, Simp
7. B 3, 4, MP
8. C 6, 7, MP
9. D 2, 8, MP

$(A \supset B) \wedge (B \supset C), C \supset D, A \vdash D$

The deduction system DS_4

Rules of inferences of DS_4

1. $(A \supset B), (B \supset C \vdash (A \supset C)$ Hypothetical Syllogism (HS)
2. $(A \supset B), (C \supset D), (A \vee C \vdash (B \vee D)$ Constructive Dilemma(CD)

Theorem 2-4-1:- $A \supset B, C \supset A \vdash C \supset B$

Proof

1) $A \supset B$ assumption
2) $C \supset A$ assumption
3) $C \supset B$ 2, 1, HS

$\therefore A \supset B, C \supset A \vdash C \supset B$

Theorem 2-4-2:- $A \supset B, A \vee C, C \supset D \vdash B \vee D$

Proof

1) $A \supset B$ assumption
2) $A \vee C$ assumption
3) $C \supset D$ assumption
4) $B \vee D$ 1, 2, 3, CD

$\therefore A \supset B, A \vee C, C \supset D \vdash B \vee D$

Theorem 2-4-3:- $D \supset (A \supset B), D \wedge C, C \supset (E \supset A) \vdash E \supset B$

Proof

1) $D \supset (A \supset B)$ assumption
2) $D \wedge C$ assumption
3) $C \supset (E \supset A)$ assumption
4) D 2, simp
5) $C \wedge D$ 2, com
6) C 5, simp
7) $E \supset A$ 3, 6, MP
8) $A \supset B$ 1, 4, MP
9) $E \supset B$ 7, 8, HS

$\therefore D \supset (A \supset B), D \wedge C, C \supset (E \supset A) \vdash E \supset B$

Theorem 2-4-4:- $A \vee B, (B \supset D) \wedge (A \supset E) \vdash \neg(D \vee E) \vee (E \vee D)$

Proof

1) $A \vee B$ *assumption*
2) $(B \supset D) \wedge (A \supset E)$ *assumption*

3) $B \supset D$ 2, simp

4) $(A \supset E) \wedge (B \supset D)$ 2, com

5) $A \supset E$ 4, simp

6) $E \vee D$ 1, 3, 5, CD

7) $(E \vee D) \vee \neg(D \vee E)$ 6, add

8) $\neg(D \vee E) \vee (E \vee D)$ 7, com

$\therefore A \vee B, (B \supset D) \wedge (A \supset E) \vdash \neg(D \vee E) \vee (E \vee D)$

Theorem 2-4-5:- $(A \supset B) \wedge C, D \supset E, C \supset D \vdash B \vee E$

Proof

1) $(A \supset B) \wedge C$ assumption

2) $D \supset E$ assumption

3) $C \supset D$ assumption

4) $A \supset B$ 1, simp

5) $C \wedge (A \supset B)$ 1, com

6) C 5, simp

7) $C \supset E$ 2, 3, HS

8) $C \vee A$ 6, add

9) $E \vee B$ 4, 7, 8, CD

10) $B \vee E$ 9, com

$\therefore (A \supset B) \wedge C, D \supset E, C \supset D \vdash B \vee E$

3- The soundness and completeness of the DSi, $1 \le i \le 4$

In this part of the paper we will prove the soundness and the completeness of the non-formal systems (LDSi), $1 \le i \le 4$.

For both systems DS_i we suggest defining a symbol (LDSi) to represent the set of all previous theorems DS_i, in Otherwise:

LDSi= { DS_i, $1 \le i \le 4$}.

3-1 Definition :(contradiction).

contradiction is a wff that is $\perp$ under any possible $\top$ assignment of truth values of the wff .

Such propositions are called un-satisfiable. Conversely, a contradiction is $\neg\top$.

3-2 Definition(soundness 1).

If LDSi is a set of theorems , and φ is a single wff , we say a deductive is sound if

$$\text{LDSi} \vdash \varphi \supset \text{LDSi} \models \varphi$$

to mean that φ may be derived from LDSi using only the rules of inference.

Remark .

Every theorem in DS_i, $1 \le i \le 4$, $1 \le i \le 4$ is $\top$

3-3 Definition a model:

A model is a deductive system consisting a set of finite assumption , and a theorem LDSi.

3-4 Definition.

If LDSi is consistent in deduction systems andif there is no wffφsuch that LDSi $\vdash\varphi$ and LDSi $\vdash\neg\varphi$. Otherwise, LDSi is D-inconsistent.

Remark.If LDSiis a tautology then (¬LDSi) is not satisfiable.

3-5 Definition.

If LDSi is deductive complete if it is deductive consistent and for every formula φ,LDSi $\vdash\varphi$ or LDSi $\vdash\neg\varphi$.

3 -6 Definition (soundness 2).

If LDSi is a set of wffs , and $\varphi$is a single wff, we say a deductive is sound if LDSi is satisfiable then LDSi is deduction consistent.

Remark.An argument is sound if and only if :

1. The argument is valid.
2. All of its premises are true.

3-7 Definition (completeness 1).

If LDSi is a set of wffs , and $\varphi$is a single wff , we say a deductive is sound if :

LDSi $\models\varphi\supset$LDSi $\vdash\varphi$.

to mean that, for every model M , if M $\models$LDSi , then M $\models\varphi$.

3-8 Definition (completeness 2).

If LDSi is a set of wffs , and $\varphi$is a single wff , we say a deductive is sound if LDSi is deduction consistent then LDSi is satisfiable.

3-9 The Completeness Theorem

An inspection of the set LDSi of of formulae shows that every member of LDSi is valid. Note that if forwffsφ and ψ,

if $\models\varphi$and $\models\varphi\supset\psi$ then $\models\psi$.

3-10 Theorem (soundness)

If LDSi $\vdash\varphi$ then LDSi $\models\varphi$.

3-11 Theorem (Godel Completeness Theorem)

If LDSi $\models\varphi$ then LDSi $\vdash\varphi$.

3-12.Proposition .

Theorems 3-11 and3-12 are equivalent.

Proof.

First, we assume that Theorem 3-11 is true and prove that Theorem 3-12 follows. Then, we assume that Theorem 3-12 is true and prove that Theorem 3-11 follows.

Suppose Theorem 3-11 is true. We want to show that Theorem 3-12 follows.

To that end, suppose that LDS iis consistent. We must show that there is a model Much that M $\models$LD Si.

LDSiis consistent. Thus, for every formula ψ ,LDSi$\nvdash$(ψ∧ ¬ψ). Thus, by the contra positive of Theorem 4.10, it follows that LDSi$\not\models$ (ψ∧ ¬ψ). That is, it is not the case that every model that makes LDSitrue also makes (ψ∧¬ψ) true. Thus, there is a model in which LDS iis true and (ψ∧¬ψ).

Thus, there is a model in which LDS iis true, as required.

Thus, Theorem 4.11 entails Theorem 4.12.

Now, suppose Theorem 4.11 holds. And suppose that LDSi $\models\varphi$. Then there is no model of LDS_i, $\neg\varphi$. Thus, by the contra positive to Theorem 4.12, LDSi , $\neg\varphi$ is not consistent. That is,

$$LDSi, \varphi \vdash (\psi \wedge \neg \psi)$$

It follows from this that

$$LDSi \vdash (\neg\varphi \subset (\psi \wedge \neg \psi)$$

Thus ,

$$LDSi \vdash (\neg (\psi \wedge \neg \psi) \subset \varphi)$$

And, since LDSi $\vdash\neg$ (ψ∧¬ψ), by modus ponens we have that

$$LDSi \vdash \varphi$$

as required. Thus, Theorem 3-12 entails Theorem 3-11.Thus, Theorem 3-11 and Theorem 3-12 are equivalent.

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