Journal Of Education Sciences
Ten Issue - September 2022

The completeness of "Hilbert – Ackermann Axiomatic System" Logic and Descriptive

Deductive Mathematical and Descriptive Logic and propositional calculus

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المستخلص :

استخدمت الباحثة في هذه الورقة أساليب وطرق أثبات استكمال نظام:
The Deductive Mathematical and Descriptive Logic and propositional calculus To Proof

The completeness of "Hilbert – Ackermann Axiomatic System"

من خلال تطبيق واستخدام الأتي:

- تقديم مفاهيم وقوانين و لغة النسق المنطقي
- تقديم أساسيات وقوانين،استخدام Deductive Logic and Descriptive
- استخدام لغة و قوانين الانسقة المنطقية.
- إثبات استكمال نظام المسلمات لـ Hilbert and Ackermann.

Background and Language

The capital letters of the alphabet, standing as propositional variables. These are atomic formulas. Conventionally, either the Latin alphabet (A , B, C) or the Greek alphabet (χ , φ, ψ) is used, but the two are not mixed.

Symbols denoting the following connectives (or logical operators):

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. (We may do with fewer operators (and thus symbols)by having some abbreviate others e.g. $P \rightarrow Q$ is equivalent to $\neg P \vee Q$.)

The left and right parentheses: (,).

The set of well-formed formulas (wffs) is recursively defined by the following rules:

1. Basis: Letters of the alphabet (usually capitalized such as A, B, φ, χ, etc.) are wffs.

2. Inductive clause I: If φ is a wff, then $\neg \varphi$ is a wff.

3. Inductive clause II: **If φ and ψ are wffs, then (φ ∧ ψ), (φ ∨ ψ), (φ → ψ), and (φ ↔ ψ) are wffs.**

4. Closure clause: **Nothing else is a wff.**

Repeated applications of these three rules permit the generation of complex wffs. For example:

1. By rule 1, A is a wff.

2. By rule 2, ¬ A is a wff.

3. By rule 1, B is a wff.

4. By rule 3, (¬ A ∨ B) is a wff.

I) Basic argument forms of the calculus

name	sequent	Description
Modus ponens	[(p→q)∧q] ├ p	if p then q ; p; therefore q
Modus Tollens	[(p → q) ∧ ¬q] ├ ¬p	if p then q; not q; therefore not p
Hypothetical Syllogism	[(p → q) ∧ (q → r)] ├ (p → r)	if p then q; if q then r; therefore, if p then r
Disjunctive Syllogism	[(p ∨ q) ∧ ¬p] ├ q	Either p or q; not p; therefore, q
Destructive Dilemma	[(p → q) ∧ (r → s) ∧ (¬q ∨ ¬s)] ├ (¬p ∨ ¬r)	If p then q; and if r then s; but either not q or not s; therefore rather not p or not r
Simplification	(p ∧ q) ├ p	p and q are true; therefore p is true
Conjunction	p, q ├ (p ∧ q)	p and q are true separately; therefore they are true conjointly
Addition	p ├ (p ∨ q)	p is true; therefore the disjunction (p or q) is true
Composition	[(p → q) ∧ (p → r)] ├ [p → (q ∧ r)]	If p then q; and if p then r; therefore if p is true then q and r are true

De Morgan's Theorem (1)	¬ (p ∧ q) ├ (¬p ∨ ¬q)	**The negation of (p and q) is equiv. to (not p or not q)**
De Morgan's Theorem (2)	¬ (p ∨ q) ├ (¬p ∧ ¬q)	**The negation of (p or q) is equiv. to (not p and not q)**
Commutation (1)	(p ∨ q) ├ (q ∨ p)	**(p or q) is equiv. to (q or p)**
Commutation (2)	(p ∧ q) ├ (q ∧ p)	**(p and q) is equiv. to (q and p)**
Association (1)	[p ∨ (q ∨ r)] ├ [(p ∨ q) ∨ r]	**p or (q or r) is equiv. to (p or q) or r**
Association (2)	[p ∧ (q ∧ r)] ├ [(p ∧ q) ∧ r]	**p and (q and r) is equiv. to (p and q) and r**
Distribution (1)	[p ∧ (q ∨ r)] ├ [(p ∧ q) ∨ (p ∧ r)]	**p and (q or r) is equiv. to (p and q) or (p and r)**
Distribution (2)	[p ∨ (q ∧ r)] ├ [(p ∨ q) ∧ (p ∨ r)]	**p or (q and r) is equiv. to (p or q) and (p or r)**
Double Negation	p ├ ¬¬p	**p is equivalent to the negation of not p**
Transposition	(p → q) ├ (¬q → ¬p)	**If p then q is equiv. to if not q then not p**
Material Implication	(p → q) ├ (¬p ∨ q)	**If p then q is equiv. to either not p or q**
Material Equivalence (1)	(p ↔ q) ├ [(p → q) ∧ (q → p)]	**(p is equiv. to q) means, (if p is true then q is true) and (if q is true then p is true)**
Material Equivalence (2)	(p ↔ q) ├ [(p ∧ q) ∨ (¬q ∧ ¬p)]	**(p is equiv. to q) means, either (p and q are true) or (both p and q are false)**
Exportation	[(p ∧ q) → r] ├ [p → (q → r)]	**from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)**
Importation	[p → (q → r)] ├ [(p ∧ q) → r]	
Tautology	p ├ (p ∨ p	**p is true is equiv. to p is true or p is true**

II) Hilbert – Ackermann Axiomatic System

In this paper the writer will illustrate the Axiomatic system of Hilbert – Ackermann denoted by (AX_{HA}) Which has eighteen theorems, This system used $\vee$, $\supset$ and $\neg$ as a primitive connectives. and $A \supset B$ as an abbreviation for $\neg A \vee B$. And we have four axiom schemas. The only rule of inference is Modus Ponens. We will give a full proof of the completeness and soundness of this system.

Historical biography Wilhelm Ackermann

Wilhelm Ackermann was born on 29 March 1896 in Schonebeck Germany. He was a mathematical logician who worked with Hilbert in Gottingen. Ackermann received his doctoral degree in 1925. He died in 24 December 1962 in Ludenscheid, Germany.

Ackermann was also the main contributor to the development of the logical system known as the epsilon calculus, originally due to Hilbert. This formalism formed the basis of Bourbaki's logic and set theory. From 1929 until 1948 he taught as a teacher at the Arnoldinum Gymnasium in Burgsteinfurt and in Luedenscheid. He was corresponding member of the Academia Wissenschaften in Gottingen, and was honorary professor at the University Munster.

Among Ackermann's later work are consistency proofs for set theory (1937), full arithmetic (1940) and type free logic (1952). Further there was a new axiomatization of set theory (1956), and a book Solvable cases of the decision problem (North Holland, 1954). [4]

III) Axiom and Theorems schemas:

$AX_{HA}1$: $A \vee A \supset A$

$AX_{HA}2$: $A \supset A \vee B$

$AX_{HA}3$: $A \vee B \supset B \vee A$

$AX_{HA}4$: $(B \supset C) \supset (A \vee B \supset A \vee C)$ [11]

Theorem 3.1 : - $A \supset B \vdash_{HA} C \vee A \supset C \vee B$.

Proof : -

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(1) $A \supset B$
assumption.

(2) $(A \supset B) \supset (C \vee A \supset C \vee B)$
$AX_{HA}4$

(3) $C \vee A \supset C \vee B$ 1 , 2 , MP

$\therefore A \supset B \vdash_{HA} C \vee A \supset C \vee B.$

Theorem 3.2:- $\vdash_{HA} (A \supset B) \supset ((C \supset A) \supset (C \supset B)).$

Proof : -

(1) $(A \supset B) \supset (\neg C \vee A \supset \neg C \vee B)$
AX_{HA}

(2) $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$ definition of implication

$\therefore \vdash_{HA} (A \supset B) \supset ((C \supset A) \supset (C \supset B)).$

Theorem 2.3:- $C \supset A, A \supset B \vdash_{HA} C \supset B.$

Proof : -

(1) $C \supset A$
assumption.

(2) $A \supset B$
assumption.

(3) $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$
Theorem 3.2

(4) $((C \supset A) \supset (C \supset B))$ 2 , 3 , MP

(5) $C \supset B$ 1 , 4 , MP

$\therefore C \supset A, A \supset B \vdash_{H} C \supset B.$

Theorem 3.4:- $\vdash_{HA} (A \supset A)$ (i.e., $\vdash_{HA} (\neg A \vee A)$)

Proof : -

(1) $((A \vee A) \supset A) \supset ((A \supset (A \vee A)) \supset (A \supset A))$
Theorem 3.2

(2) $(A \vee A) \supset A$
$AX_{HA}1$

(3) $(A \supset (A \vee A)) \supset (A \supset A)$ **1 , 2 , MP**

(4) $A \supset (A \vee A)$
$AX_{HA}2$

(5) $A \supset A$ **3 , 4 , MP**

$\therefore \vdash_{HA} (A \supset A)$

Theorem 3.5:- $\vdash_{HA} (A \vee \neg A)$

Proof : -

(1) $A \supset A$ **Theorem 3.4**

(2) $\neg A \vee A$ **Definition of implication**

(3) $(\neg A \vee A) \supset (A \vee \neg A)$
$AX_{HA}3$

(4) $(A \vee \neg A)$ **2 , 3 , MP**

$\therefore \vdash_{HA} (A \vee \neg A)$

Theorem 3.6:- $\vdash_{HA} A \supset \neg\neg A$

Proof : -

(1) $\neg A \vee \neg\neg A$ **Theorem 3.5**

(2) $A \supset \neg\neg A$ **Definition of implication**

$\therefore \vdash_{HA} A \supset \neg\neg A$

Theorem 3.7 :- $\vdash_{HA} \neg\neg A \supset A$

Proof :-

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(1) ¬A ⊃ ¬¬¬A Theorem 3.6

(2) (¬A ⊃ ¬¬¬A) ⊃ ((A ∨ ¬ A) ⊃(A ∨ ¬¬¬A)) $AX_{HA}4$

(3) (A ∨ ¬ A) ⊃(A ∨ ¬¬¬A) 1 , 2 , MP

(4) (A ∨ ¬ A) Theorem 3.5

(5) A ∨ ¬¬¬A 3 , 4 , MP

(6) (A ∨ ¬¬¬A) ⊃ (¬¬¬A ∨ A) $AX_{HA}3$

(7) ¬¬¬A ∨ A 5 , 6 , MP

(8) ¬¬A ⊃ A 7 , Definition of implication

∴ $\vdash_{HA}$ ¬¬A ⊃ A

Theorem 3.8:- $\vdash_{HA}$ ¬B ⊃ (B ⊃C)

Proof : -

(1) ¬B ⊃ (¬B ∨ C) $AX_{HA}2$

(2) ¬B ⊃ (B ⊃ C) Definition of implication

∴ $\vdash_{HA}$ ¬B ⊃ (B ⊃C)

Theorem 3.9:- $\vdash_{HA}$ A ∨ (B ∨ C) ⊃ ((B ∨ (A ∨ C)) ∨ A)

Proof : -

(1) C ⊃ (C ∨ A) $AX_{HA}2$

(2) (C ∨ A) ⊃ (A ∨ C) $AX_{HA}3$

(3) C ⊃ (A ∨ C) 1 , 2 , Theorem 3.3

(4) $(C \supset (A \vee C)) \supset ((B \vee C) \supset (B \vee (A \vee C)))$ **$AX_{HA}4$**

(5) $(B \vee C) \supset (B \vee (A \vee C))$ **3 , 4 , MP**

(6) $((B \vee C) \supset (B \vee (A \vee C))) \supset ((A \vee (B \vee C)) \supset (A \vee (B \vee (A \vee C))))$ **$AX_{HA}4$**

(7) $(A \vee (B \vee C)) \supset (A \vee (B \vee (A \vee C)))$ **5, 6 , MP**

(8) $(A \vee (B \vee (A \vee C))) \supset ((B \vee (A \vee C)) \vee A)$ **$AX_{HA}3$**

(9) $A \vee (B \vee C) \supset ((B \vee (A \vee C)) \vee A)$ **7, 8 , Theorem 3.3**

$\therefore \vdash_{HA} A \vee (B \vee C) \supset ((B \vee (A \vee C)) \vee A)$

Theorem 3.10:- $\vdash_{HA} ((B \vee (A \vee C)) \vee A) \supset (B \vee (A \vee C))$

Proof : -

(1) $(A \vee C) \supset ((A \vee C) \vee B)$ **$AX_{HA}2$**

(2) $((A \vee C) \vee B) \supset (B \vee (A \vee C))$ **$AX_{HA}3$**

(3) $(A \vee C) \supset (B \vee (A \vee C))$ **1,2, Theorem 3.3**

(4) $((A \vee C) \supset (B \vee (A \vee C))) \supset ((A \supset (A \vee C)) \supset (A \supset (B \vee (A \vee C))))$ **Theorem3.2**

(5) $(A \supset (A \vee C)) \supset (A \supset (B \vee (A \vee C)))$ **3, 4 ,MP**

(6) $A \supset (A \vee C)$ **$AX_{HA}2$**

(7) $A \supset (B \vee (A \vee C))$ **5 , 6 , MP**

(8) $(A \supset (B \vee (A \vee C))) \supset (((B \vee (A \vee C)) \vee A) \supset ((B \vee (A \vee C)) \vee (B \vee (A \vee C))))$ **$AX_{HA}4$**

(9) $((B \vee (A \vee C)) \vee A) \supset ((B \vee (A \vee C)) \vee (B \vee (A \vee C)))$ **7, 8, MP**

(10) $((B \vee (A \vee C)) \vee (B \vee (A \vee C))) \supset (B \vee (A \vee C))$ **$AX_{HA}1$**

(11) $((B \vee (A \vee C)) \vee A) \supset (B \vee (A \vee C))$ **9, 10,Theorem 3.3**

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$\therefore \vdash_{HA} ((B \vee (A \vee C)) \vee A) \supset (B \vee (A \vee C))$

Theorem 3.11:- $\vdash_{HA} (A \vee (B \vee C)) \supset (B \vee (A \vee C))$

Proof : -

(1) $A \vee (B \vee C) \supset ((B \vee (A \vee C)) \vee A)$ theorem 3.9

(2) $((B \vee (A \vee C)) \vee A) \supset (B \vee (A \vee C))$ theorem 3.10

(3) $A \vee (B \vee C) \supset (B \vee (A \vee C))$ theorem 3.3

$\therefore \vdash_{HA} (A \vee (B \vee C)) \supset (B \vee (A \vee C))$

Theorem 3.12:- $\vdash_{HA} (A \supset (B \supset C)) \supset (B \supset (A \supset C))$

Proof : -

(1) $(\neg A \vee (\neg B \vee C)) \supset (\neg B \vee (\neg A \vee C))$ Theorem 3.11

(2) $(A \supset (\neg B \vee C)) \supset (B \supset (\neg A \vee C))$ definition of implication

(3) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ definition of implication

$\therefore \vdash_{HA} (A \supset (B \supset C)) \supset (B \supset (A \supset C))$

Theorem 3.13 : - $\vdash_{HA} (A \supset B) \supset (\neg B \supset \neg A)$

Proof : -

(1) $B \supset \neg\neg B$ Theorem 3.6

(2) $(B \supset \neg\neg B) \supset ((\neg A \vee B) \supset (\neg A \vee \neg\neg B))$ $AX_{HA}4$

(3) $(\neg A \vee B) \supset (\neg A \vee \neg\neg B)$ 1 , 2 , MP

(4) $(\neg A \vee \neg\neg B) \supset (\neg\neg B \vee \neg A)$ $AX_{HA}3$

(5) $((\neg A \vee \neg\neg B) \supset (\neg\neg B \vee \neg A)) \supset$
$(((\neg A \vee B) \supset (\neg A \vee \neg\neg B)) \supset ((\neg A \vee B) \supset (\neg\neg B \vee \neg A)))$ theorem3.2

(6) $((\neg A \vee B) \supset (\neg A \vee \neg\neg B)) \supset ((\neg A \vee B) \supset (\neg\neg B \vee \neg A))$ 4, 5, MP

(7) $(\neg A \vee B) \supset (\neg\neg B \vee \neg A)$ 3, 6 ,MP

(8) $(A \supset B) \supset (\neg B \supset \neg A)$ definition of implication twice

$\therefore \vdash_{HA} (A \supset B) \supset (\neg B \supset \neg A)$

Theorem 3.14:- $\vdash_{HA} (C \supset A) \supset ((A \supset B) \supset (C \supset B))$

Proof : -

(1) $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$ Theorem 3.2

(2) $(A \supset B) \supset ((C \supset A) \supset (C \supset B)) \supset ((C \supset A) \supset ((A \supset B) \supset (C \supset B)))$ Theorem 3.12

(8) $(C \supset A) \supset ((A \supset B) \supset (C \supset B))$ 1 , 2 , MP

$\therefore \vdash_{HA} (C \supset A) \supset ((A \supset B) \supset (C \supset B))$

Theorem 3.15:- $A \supset (B \supset C), A \supset B \vdash_{HA} A \supset (A \supset C)$

Proof : -

(1) $A \supset (B \supset C)$ assumption

(2) $A \supset B$ assumption

(3) $(A \supset B) \supset ((B \supset (A \supset C)) \supset (A \supset (A \supset C)))$ theorem 3.14

(4) $(B \supset (A \supset C)) \supset (A \supset (A \supset C))$ 2 , 3 , MP

(5) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ Theorem 3.12

(6) $B \supset (A \supset C)$ 1 , 5 , MP

(7) $A \supset (A \supset C)$ 4 , 6 , MP

$\therefore A \supset (B \supset C) , A \supset B \vdash_{HA} A \supset (A \supset C)$

Theorem 3.16:- $A \supset (B \supset C) , A \supset B \vdash_{HA} A \supset C$

Proof : -

(1) $A \supset (B \supset C)$ assumption

(2) $A \supset B$ assumption

(3) $\neg A \supset (A \supset C)$ theorem 3.8

(4) $(\neg A \supset (A \supset C)) \supset (\neg (A \supset C) \supset \neg\neg A)$ theorem 3.13

(5) $(\neg A \supset (A \supset C)) \supset \neg\neg A$ 3 , 4 , MP

(6) $\neg\neg A \supset A$ theorem 3.6

(7) $\neg(A \supset C) \supset A$ 5 , 6 , theorem 3.3

(8) $A \supset (A \supset C)$ 1 , 2 , theorem 3.14

(9) $\neg(A \supset C) \supset (A \supset C)$ 7 , 8 , theorem 3.3

(10) $(A \supset C) \supset \neg\neg (A \supset C)$ theorem 3.6

(11) $\neg(A \supset C) \supset \neg\neg (A \supset C)$ 9 , 10 , theorem 3.3

(12) $\neg\neg(A \supset C) \vee \neg\neg(A \supset C)$ definition of implication

(13) $(\neg\neg(A \supset C) \vee \neg\neg(A \supset C)) \vee \neg\neg(A \supset C)$ $AX_{HA}3$

(14) $\neg\neg(A \supset C)$ 12 , 13 , MP

(15) $\neg\neg(A \supset C) \supset (A \supset C)$ Theorem 3.7

(16) $A \supset C$ 14 , 15 , MP

$\therefore A \supset (B \supset C) , A \supset B \vdash_{HA} A \supset C$

Theorem 3.17:- If $\Gamma, A \vdash_{HA} B$ Then $\Gamma \vdash_{HA} A \supset B$ (Deduction theorem) [1]

Proof : -

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Let B_0 , B_1 , … , B_{n-1} be a proof of B from Γ , A

We must show by induction that for every i < n , $A \supset B_i$ is provable from Γ.

Case (1)

B_i is a logical axiom or a member of Γ , in this case B_i can be used in proving $A \supset B_i$ from Γ.

Case (2)

B_i = A in This case $A \supset B_i = A \supset A$ and by theorem 3.4 ($A \supset A$) is provable in the deductive system.

Now suppose that the deduction of B_i from $\Gamma \cup \{A\}$ is a sequence with n members, where n > 1, and that the proposition hold for all wffs. C which can be deduced from $\Gamma \cup \{A\}$ with fewer than n members. This time there are for cases

Case 1:
Case 2: } as two cases above
Case (3)

Bi is obtained from tow formulas say $C \supset Bi$ and C.

by the induction hypothesis $\vdash_{HA} A \supset (C \supset B_i)$, $\vdash_{HA} (A \supset C)$

(1) $A \supset (C \supset B_i)$ assumption

(2) $A \supset C$ assumption

(3) $A \supset B_i$ theorem 3.16

$\therefore \Gamma \vdash_{HA} A \supset B$

Theorem 3.18:- $B \supset A , \neg B \supset A \vdash_{HA} A$

Proof : -

(1) $B \supset A$ assumption

(2) $\neg B \supset A$ assumption

(3) $(B \supset A) \supset (\neg A \supset \neg B)$ theorem 3.12

(4) $\neg A \supset \neg B$ 1 , 3 , MP

(5) $\neg A \supset A$ 2 , 4 , theorem 3.3

(6) $(\neg A \supset A) \supset (\neg A \supset \neg\neg A)$ theorem 3.13

(7) $(\neg A \supset \neg\neg A)$ 5 , 6 , MP

(8) $\neg\neg A \vee \neg\neg A$ 7 , Definition of Implication

(9) $(\neg\neg A \vee \neg\neg A) \supset \neg\neg A$ $AX_{HA}1$

(10) $\neg\neg A$ 9 , 8 , MP

(11) $\neg\neg A \supset A$ Theorem 3.7

(12) A 10 , 11 , MP

$\therefore B \supset A , \neg B \supset A \vdash_{HA} A$

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